APPROXIMATE DETERMINATION OF THE GAS POPULATION INVERSION AND GAIN COEFFICIENT DURING ADIABATIC EXPANSION IN A NOZZLE

G. V. Gembarzhevskii

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Approximate analytical expressions are obtained herein for the power and weak-signal gain coefficient of a gasdynamic laser under more general assumptions than have been used earlier. Thus, the following assumptions were made in [1] to obtain a simple theoretical estimate of the weak-signal gain coefficient: the population of the third CO_2 mode is constant everywhere in the gas stream and equal to the population in the nozzle throat, and the population of the other two CO_2 modes is in equilibrium with translational motion. Approximately the same results have been obtained in [2], but as a result of an assumption about an infinitely high rate of gas cooling in the nozzle as compared with the vibrational relaxation rate. Hence [2] permits finding the domain of applicability of the results obtained in [1], which turns out to be bounded with respect to low stagnation temperature and pressure.

A similarity law for the gain coefficient of a gasdynamic laser in the domain of comparatively high stagnation pressures has been proposed in [3]. The solution of a model problem on vibrational relaxation with constant reaction rates permitted the similarity law to be obtained only to the accuracy of an unknown function of the stagnation temperature, the adiabatic gas index, and the nozzle configuration. A numerical analysis of the system of kinetic and gasdynamic equations for the nozzle [4-6] is needed to determine this function.

The domain of applicability of the results obtained [1-3] is bounded substantially because of neglecting the change in vibrational relaxation rate along the nozzle: thus, an upper bound to the stagnation parameters is required in [1, 2], and the nozzle configuration, gas stagnation temperature, and adiabatic index are fixed in [3].

It is proposed to obtain analytic expressions herein for the dependences of the power and gain coefficient of a gasdynamic laser on the nozzle shape and size, and also, by taking account of the time dependences of the kinetic equation coefficients, the gas stagnation parameters and composition which would be valid in a broad range of variation of these parameters.

Let us take the ordinary assumptions used in numerical computations [4-6]: there exist vibrational temperatures T_i of the individual mixture modes (the subscript $i=1, 2, 3$ refers to the CO_2 modes, and $i=4$, to nitrogen), the rotational and translational temperatures equal T, the gasdynamic flow is quasi-one-dimensional, and viscosity and heat conduction effects are neglected. Moreover, we take two more barely essential assumptions: the characteristic temperature of the third CO_2 mode equals the characteristic temperature of the fourth mode, i.e., of the nitrogen $\theta_3 = \theta_4$, the rate of exchange between the first and second CO_2 modes is greater than the gas cooling rate, i.e., $Y_1 = Y_2^2 \exp(-78/T) \approx Y_2^2$, where $Y_i = \exp(-\theta_i/T_i)$ is a quantity governing the population of the mode i, and θ_i is the characteristic temperature of the mode i. This assumption is used in [3, 6], and it is shown in [5] that it is satisfied to a high degree of accuracy even at low pressures.

And, finally, the main assumption is the following: The time dependence of the gasdynamic quantities in the kinetic equation coefficients can be approximated by a dependence for a gas flow with a constant adiabatic index γ. Indeed, the data presented in [6] permit the assertion that the accuracy of the population inversion calculation is within $\pm 25\%$ limits when this latter assumption is used. The error in the

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computation will be lower, the smaller the initial store of vibrational energy, i.e., the lower the stagnation temperature and the higher the relative concentrations of helium Ψ_3 and nitrogen Ψ_2.

Using the above-mentioned assumptions, we obtain the system

$$dY_2 / dX_2 = A_{21}(Y_2 - X_2) - mA_{31}(Y_3 - Y_2^3 X_2^{-3} X_3) \quad (1)$$

$$dY_3 / dX_3 = A_{31}(Y_3 - Y_2^3 X_2^{-3} X_3) - A_{32}(Y_4 - X_3) \quad (2)$$

$$dY_4 / dX_4 = A_{41}(Y_4 - X_4) - rA_{32}(Y_3 - Y_4) \quad (3)$$

where the following notation has been introduced: X_i are equilibrium values of Y_i:

$$X_i = \exp(-\theta_i / T), \qquad A_{ij} = N_* W_{ij} R_{ij} H^{-1} X_i^{-1}$$

$$W_{21} = 0.5\theta_2^{-1} \sum_{\alpha=1}^{3} P_{21}^{\alpha} \Psi_\alpha \sigma_\alpha^1 U_\alpha^1 / C$$

$$W_{31} = \theta_3^{-1} \sum_{\alpha=1}^{3} P_{31}^{\alpha} \Psi_\alpha \sigma_\alpha^1 U_\alpha^1 / C$$

$$W_{41} = \theta_4^{-1} \sum_{\alpha=1}^{3} P_{41}^{\alpha} \Psi_\alpha \sigma_\alpha^2 U_\alpha^2 / C$$

$$W_{32} = \theta_3^{-1} P_{32}^2 \Psi_2 \sigma_2^1 U_2^1 / C$$

N_* is the value of the gas density in the nozzle throat, Ψ_α is the relative concentration of the substance α (the subscript α takes on the values 1, 2, 3, denoting carbon dioxide gas, nitrogen, and helium, respectively), σ_α^1 is the collision cross section between the CO_2 and α molecules according to the data in [7], σ_α^2 is the collision cross section between the N_2 and α molecules according to [7], U_α^1 is the mean relative velocity of the molecules α and CO_2, U_α^2 is the mean relative velocity of the molecules α and N_2, C is the speed of sound, and P_{ij}^{α} (P_{32}^2) are the probabilities of the corresponding vibrational transitions for a single collision with the molecule α (N_2). In conformity with [5, 7], the following values are taken here:

$$P_{21}^1 = 450T^{-1/3} \exp(-110T^{-1/3} + 100T^{-2/3})$$
$$P_{21}^2 = 3.0P_{21}^1, \quad P_{21}^3 = 10T^{-1/2} \exp(-50T^{-1/3})$$
$$P_{31}^1 = 2.5 \exp(-80T^{-1/3} + 60T^{-2/3}), \quad P_{31}^2 = 0.4P_{31}^1$$
$$P_{31}^3 = 0.16P_{31}^1, \quad P_{41}^{\alpha} = 1.4 \cdot 10^5 \exp(-280T^{-1/3})$$
$$P_{32}^2 = 4 \cdot 10^{-9}(1100 - T)^2 + 6 \cdot 10^{-4}$$
$$R_{21} = (1 - Y_2)^4 (1 - Y_2^2)(1 - Y_3)[1 + 2Y_2(1 + Y_2)^{-2}]^{-1}$$
$$R_{31} = (1 - Y_2)^2 (1 - Y_2^2)(1 - Y_3)^3, \quad R_{41} = (1 - Y_4)^3$$
$$R_{32} = (1 - Y_2)^2 (1 - Y_2^2)(1 - Y_3)^3 (1 - Y_4)$$
$$m = 5.25(1 - Y_2)^2 (1 - Y_3)^{-2}[1 + 2Y_2(1 + Y_2)^{-2}]^{-1} \exp(-2420 \cdot T^{-1})$$
$$r = \Psi_1 \Psi_2^{-1} (1 - Y_4)^2 (1 - Y_3)^{-2}$$
$$H = \frac{dS}{dl} \frac{[1 + 2k(1 - t)]^2}{S_*(1 - t)t^{1/2} Tk(1 + 2k)}$$

where $S = S(l)$ is the nozzle section at a distance l from the throat, S_* is the nozzle section in the throat, T_* is the translational temperature in the throat, and $t = T/T_*$, $k = (\gamma - 1)^{-1}$, γ is the mixture adiabatic index.

Let us assume the domain of "freezing" of the third and the two other CO_2 modes to be separated, then $Y_3 - Y_2^3 X_2^{-3} X_3 \simeq Y_3 - X_3$. It is shown below that this is actually so in the interesting cases from the viewpoint of obtaining population inversions.

Considering the system (1)-(3), let us note that the coefficients A depend strongly on X so that $AX \gg 1$ in the nozzle throat, while $AX \ll 1$ at the end of the nozzle, i.e., the gas cooling rate is initially less than the vibrational relaxation rate, and then the picture reverses. Assuming $dY/dX \approx Y/X$ (a similar assumption was used in [8]), the solution of the system can be found in both the limit cases $AX \gtrless 1$. Finally, by merging the solutions at the point X^+ where $AX^+ = 1$, we obtain the frozen values of Y^+. The initial conditions do not influence Y^+ if the condition $AX \gg 1$.

The specific results are the following.

1°. The total rate of vibrational relaxation of N_2 and the third CO_2 mode is determined by the transition A_{31}, i.e., let $A_{31}(X^+_{231}) < rA_{32}(X^+_{331})$ where X^+_{331} is defined as the root of the equation $X_3 A_{31}(X_3) = 1$, then

$$Y_3^+ \simeq Y_4^+ \simeq \left[\left(\frac{2r+1}{rA_{31}} + \frac{1}{r(r+1)A_{32}}\right)\left(1 + \frac{A_{41}}{rA_{31}} + \frac{A_{41}}{rA_{32}}\right)^{-1}\right]_{X_3=X_{331}^+} \tag{4}$$

The expression in the square brackets here and in (5), (6) is taken at the points indicated at the brackets.

2°. The transition A_{32} is the limiting transition. Let

$$rA_{32}(X_{332}^+) < A_{31}(X_{332}^+), \quad A_{41}(X_{332}^+) < rA_3(X_{332}^+), \quad X_{332}^{++} < X_{331}^+ < X_{332}^+$$

where X_{331}^+ has been defined above, X_{332}^{++} and X_{332}^+ are, respectively, the smaller and larger roots of the equation $X_3 rA_{32}(X_3) = 1$. In this case

$$\left[\left(\frac{1}{rA_{31}} + \frac{2}{r(r+1)A_{32}}\right)\left(1 + \frac{A_{41}}{rA_{31}} + \frac{A_{41}}{rA_{32}}\right)^{-1}\right]_{X_3=X_{332}^+} \leqslant Y_3^+ \simeq$$
$$\simeq Y_4^+ \leqslant \left[\left(\frac{1+r}{rA_{31}} + \frac{2}{r(r+1)A_{32}}\right)\left(1 + \frac{A_{41}}{rA_{31}} + \frac{A_{41}}{rA_{32}}\right)^{-1}\right]_{X_3=X_{332}^+} \tag{5}$$

In both the above-mentioned cases, we obtain for the frozen value of Y_2^+:

$$Y_2^+ = [2A_{21}^{-1}(1 + mA_{31}Y_3^+)]_{X_3=X_{32}^+} \tag{6}$$

where X_2^+ is defined by the equation $X_2 A_{21}(X_2) = 1$.

Let us note the following important circumstance: The frozen population of the first and second CO_2 modes $Y_1^+ = (Y_2^+)^2$ can be made arbitrarily small because of selecting the nozzle shape $S_*^{-1}\, dS/dl$ below the section where the third CO_2 mode is frozen ($X_3 \approx X_3^+$) if the inequality $W_{21}R_{21} \gg W_{31}R_{31}$ is satisfied in this region of the nozzle. For this, 30-40% of helium in the mixture turns out to be sufficient according to the values taken for the probabilities of the processes. The assumption about separation of the freezing domains of the third and the other two CO_2 modes is thereby supported.

Taking account of the above relative to the populations of the first and second CO_2 modes, let us consider the maximum power P which can be extracted from the gas mixture in the form of coherent radiation to be determined by the power stored in the third CO_2 mode and in the nitrogen, i.e.,

$$P = h(\nu_3 - \nu_1) S_* U_* N_* [\Psi_1 Y_3^+ (1 - Y_3^+)^{-1} + \Psi_2 Y_4^+ (1 - Y_4^+)^{-1}] \tag{7}$$

where $h(\nu_3 - \nu_1)$ is the radiation quantum energy, and $S_* U_* N_*$ is the gas discharge.

Optimal conditions to achieve maximum power can be mentioned by analyzing (4), (5), (7) (and considering $S = S(l)$, T_*, X_3^+, Ψ_1, $\Psi_1 + \Psi_2$ the independent variables).

1. The angle β between the axis and the generatrix of an axisymmetric nozzle, or half the angle between the planes of a wedge nozzle should be selected as great as possible since $P \sim \tan\beta$ (that part of the nozzle where the third CO_2 mode is frozen, i.e., $X_3 \approx X_3^+$ is kept in mind).

2. Since $P \sim S^{1/2}$ for an axisymmetric nozzle, then several rather than one nozzle should be used for a fixed total critical section (in conformity with the recommendations in [4]), by simultaneously increasing the density in the critical section in proportion to $S_*^{-1/2}$ (then all the remaining parameters will become invariant, particularly the Reynolds number). The power is proportional to the width for a wedge nozzle.

3. The critical temperature should be selected as high as possible since $P \sim T^{1+k/2}$ for an axisymmetric and $P \sim T_*$ for a wedge nozzle. Only the dissociation process sets the upper bound for T_*.

4. The power depends on the total relative concentrations of CO_2 and N_2 not more strongly than $P \sim \Psi_1 + \Psi_2$. The quantity $\Psi_1 + \Psi_2$ is limited by the relative He concentration required for effective deactivation of the first and second CO_2 modes ($\Psi_1 + \Psi_2 \leq 0.7$).

5. The dependence of P on x_{331}^+ for 1° exhibits a shallow maximum in the neighborhood of $X_{331}^+ = 2 \cdot 10^{-2}$ and $2 \cdot 10^{-1}$ for the axisymmetric and wedge nozzles, respectively.

6. In case 1° the optimal value is $r = A_{31}(X_{331}^+)/A_{32}(X_{331}^+)$; $P \sim \Psi_1^{-1}$.

Now, let us consider the weak-signal gain coefficient of a gasdynamic laser g. Let us note that the gain coefficient is not of greatest interest so much as the optical thickness at the nozzle width gd defined by the equality

$$\Delta I / I = \exp(gd) - 1$$

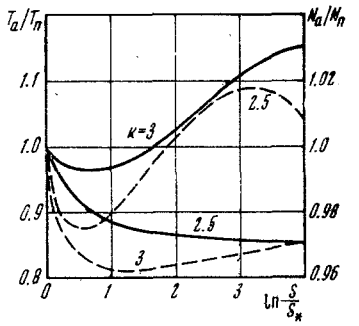

Fig. 1

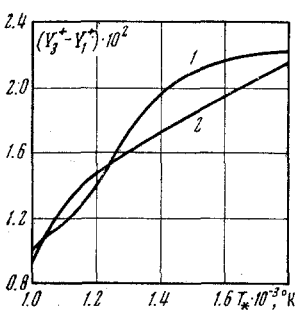

Fig. 2

where $\Delta I/I$ is the relative increase in the light intensity during passage through a layer of optically active medium of thickness d equal to the nozzle width.

It turns out that both the optical thickness gd and the gain coefficient as a function of the distance along the nozzle reach a maximum at the point where the density drops to the value $N_0 \approx 2 \cdot 10^{17}$ cm^{-3} (i.e., where the Lorentz line shape is replaced by the Doppler shape). Here g is to be understood as the gain coefficient at the center of the vibrational-rotational line, where the rotational quantum number J is selected optimal at the given temperature. The deduction presented above is valid if the freezing of the first and second CO_2 modes terminates earlier than the gas density drops to the value $N_0 \simeq 2 \cdot 10^{17}$ cm^{-3}. This can always be achieved by increasing the gas density N_* and diminishing the radius of the critical section of the axisymmetric nozzle or the height of the wedge nozzle in proportion to N_*^{-1}, where all the remaining parameters remain unchanged.

The maximum gain coefficient max g and the maximum value of light amplification over the width of an axisymmetric nozzle max gd are determined by the formulas

$$\max g \simeq a\Psi_1 (Y_3^+ - Y_1^+)(1 - Y_1^+)(1 - Y_2^+)^2 (1 - Y_3^+)\,[T(N_0)]^{-1} \tag{8}$$

$$\max gd \simeq \frac{2a\Psi_1 (Y_3^+ - Y_1^+)(1 - Y_1^+)(1 - Y_2^+)^2 (1 - Y_3^+) S_*^{1/2}}{\pi^{1/2} T(N_0)\,[t(N_0)]^{k/2}\,[1 + 2k - 2kt(N_0)]^{1/4}} \tag{9}$$

where $a = 3 \cdot 10^2$, cm^{-1} deg.

By analyzing (4), (5), (8), (9) [and considering $S = S(l)$, T_*, X_3^+, Ψ_1, $(\Psi_1 + \Psi_2)$ the independent variables], we obtain the following conditions for achieving the maximum gain in the nozzle width.

1. The angle β should be selected as large as possible since $\max gd \sim (\tan\beta)^{1/2+1/k}$, $\max g \sim (\tan\beta)^{1/k}$.

2. For an axisymmetric nozzle $\max gd \sim S^{(k-2)/4k}$. Hence, several, rather than one, axisymmetric nozzles should be used for a fixed total critical section. The width d in a wedge nozzle should be increased, and the height at the critical section h_* should be diminished since $(\max g)\, d \sim dh^{-1/k}$.

3. For an axisymmetric nozzle T_* should be selected as high as possible since $gd \sim T^{(k^2-k+2)/4k}$. For a wedge nozzle the maximum gain is achieved at a temperature determined by the expression

$$T_* \simeq -\theta_3 / \ln X_3^+$$

(i.e., freezing of the third CO_2 mode should occur in the nozzle throat).

4. The condition to achieve maximum power corresponds to Sec. 4.

5. The optimal value for a wedge nozzle is $X_{331}^+ = 3 \cdot 10^{-1}$ for 1°, while for an axisymmetric nozzle X_{331}^+ should be selected as large as possible by taking into account that the dissociation process sets the upper bound for X_{331}^+. The maximum in the dependence of the optical thickness on X_{331}^+ is barely defined (shallow).

6. The optimal value for an axisymmetric nozzle in 1° is $r = A_{31}(X_{331}^+)/A_{32}(X_{331}^+)$, while the optimal value of r for a wedge nozzle is infinite ($\Psi_2 = 0$).

7. We obtain the following result by comparing the efficiency of a wedge and a number of axisymmetric nozzles. For the fixed parameters ΣS_*, $\tan\beta$, T_*, X_3^+, $\Psi_1 + \Psi_2$, N_* (ΣS_* is the total critical section), the ratio between the maximal optical thicknesses of the wedge and a number of axisymmetric nozzles is given by the expression

$0.5\pi\,[-T(N_0)\ln X_3^+ / \theta_3]^{k/2}$

The estimate obtained above for the power and gain coefficient was verified by a numerical computation published in [5]. The results of the comparison are presented in the graphs. The solid lines in Fig. 1 represent the dependence of the ratio between the translational temperature computed analytically (for k = 2.5, 3) and the temperature computed numerically T_a/T_n on the logarithm of the relative nozzle cross section $\ln S/S_*$, and the dashes are the dependence of the ratio between the densities computed analytically (k = 2.5, 3) and numerically N_a/N_n on the same argument. The mixture composition is 10% CO_2 + 90% N_2, the temperature in the critical section is $T_* = 1290°K$, and the density is $N_* = 4 \cdot 10^{19}$ cm^{-3}. Figure 2 presents the dependence of the difference $Y_3^+ - Y_1^+$ computed analytically for k = 2 (curve 1) and numerically (curve 2) on the temperature in the critical section, where the density varies negligibly, and the mixture composition is: ($N_* \approx 4 \cdot 10^{19}$ cm^{-3}) 10% CO_2 + 40% N_2 + 50% He.

The results presented permit the assertion that the accuracy of the analytical computation is within 15% limits in the temperature, within 4% in the density, within 15% in the population intensity.

In principle, the results obtained are also applicable to other systems besides $CO_2 + N_2 + He$.

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